

## SPARSITY METHODS FOR NETWORKED CONTROL .

MASAAKI NAGAHARA

ABSTRACT. In this presentation, we introduce sparsity methods for networked control systems and show the effectiveness of sparse control. In networked control, efficient data transmission is important since transmission delay and error can critically deteriorate the stability and performance. We will show that this problem is solved by sparse control designed by recent sparse optimization methods.

## 1. INTRODUCTION

Sparsity methods, called *compressed sensing* or *sparse representation*, have been recently introduced in signal processing [4]. The methods are also applied to communications, see a survey paper [7].

A sparse vector is a vector that has very few non-zero entries compared with the vector size. Compressed sensing takes advantage of sparsity property of signals in some domain (e.g. the Fourier domain), for efficiently reconstructing such signals from very few measurements by finding a solution of underdetermined linear equations. To solve such underdetermined linear equations, one can adopt an  $L^1$ -norm minimization [3] to achieve the sparsity, or a matching pursuit to find a sparse solution in a greedy way [8].

Sparsity methods are very recently applied to solving problems in control systems such as predictive networked control [9, 11], hands-off control [10], actuator scheduling [1], and security in cyber-physical systems [5], to name a few. In this presentation, we introduce sparsity methods for networked control systems and discuss the effectiveness of the sparse control.

## 2. PROBLEM FORMULATION

We here consider linear and time-invariant (LTI) plant models of the form

$$(1) \quad \frac{d\mathbf{x}(t)}{dt} = A\mathbf{x}(t) + B\mathbf{u}(t), \quad t \in [0, T],$$

where  $\mathbf{x}(t) = [x_1(t), \dots, x_n(t)]^\top \in \mathbb{R}^n$  is the state,  $\mathbf{u}(t) = [u_1(t), \dots, u_m(t)]^\top \in \mathbb{R}^m$  is the control input, and  $T \in (0, \infty)$  is the length of the control horizon.

The control  $\{\mathbf{u}(t) : t \in [0, T]\}$  is chosen to drive the state  $\mathbf{x}(t)$  from a given initial state  $\mathbf{x}(0) = \mathbf{x}_0$  to the origin in time  $T$ , that is,  $\mathbf{x}(T) = \mathbf{0}$ . Also, the control  $\mathbf{u}(t)$  is constrained in magnitude by

$$(2) \quad \|\mathbf{u}(t)\|_\infty \leq 1, \quad \forall t \in [0, T].$$

We call a control  $\{\mathbf{u}(t) : t \in [0, T]\}$  *admissible* if it satisfies (2) and the resultant state  $\mathbf{x}(t)$  from (1) satisfies boundary conditions  $\mathbf{x}(0) = \mathbf{x}_0$  and  $\mathbf{x}(T) = \mathbf{0}$ . We

---

*Key words and phrases.* networked control, sparsity, optimal control, hands-off control.

SUBMITTED TO IEICE SMARTCOM2014

denote by  $\mathcal{U}$  the set of all admissible controls. Among all admissible controls in the set  $\mathcal{U}$ , we consider a control that maximizes the time interval over which the control  $\mathbf{u}(t)$  is exactly zero. Such a control is called a *maximum hands-off control*, the problem of which is described as follows:

**Problem 1** (Maximum Hands-Off Control). *Find an admissible control  $\{\mathbf{u}(t) : t \in [0, T]\} \in \mathcal{U}$  that minimizes*

$$(3) \quad J_0(\mathbf{u}) \triangleq \sum_{i=1}^m \lambda_i \|u_i\|_{L^0} = \sum_{i=1}^m \lambda_i \int_0^T \phi(u_i(t)) dt,$$

where  $\lambda_1 > 0, \dots, \lambda_m > 0$  are given weights and

$$\phi(u) \triangleq \begin{cases} 1, & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases}$$

This problem is quite hard to solve since the cost function is highly nonlinear and non-convex. To overcome the non-convexity, we introduce the following  $L^1$ -optimal control problem:

**Problem 2** ( $L^1$ -Optimal Control). *Find an admissible control  $\{\mathbf{u}(t) : t \in [0, T]\} \in \mathcal{U}$  that minimizes*

$$(4) \quad J_1(\mathbf{u}) \triangleq \sum_{i=1}^m \lambda_i \|u_i\|_{L^1} = \sum_{i=1}^m \lambda_i \int_0^T |u_i(t)| dt.$$

This problem is a classical  $L^1$  optimal control problem, also known as fuel-optimal control problem, and can be easily solved [2, Chap. 8]. Moreover, the following theorem shows the equivalence between the two control problems [10]:

**Theorem 1.** *Assume that  $(A, B)$  is controllable<sup>1</sup>. Assume also that Problem 2 has at least one solution<sup>2</sup>. Then the set of the solutions of Problem 1 (maximum hands-off control) is equivalent to the set of the solutions of Problem 2 ( $L^1$ -optimal control).*

### 3. NETWORKED CONTROL

Let us assume that the conditions in Theorem 1 hold. Then, the maximum hands-off control takes only 3 values,  $\{-1, 0, 1\}$ , and the value changes discontinuously. This property, called “bang-bang control,” benefits networked control systems since the control value can be represented in only 2 bits. Moreover, the number of switching times is bounded as shown in the following proposition:

**Proposition 1.** *Assume that  $(A, B)$  is controllable and  $A$  is nonsingular. Let  $\omega$  be the largest imaginary part of the eigenvalues of  $A$ . Then, the maximum hands-off control is a piecewise constant signal, with values  $-1$ ,  $0$ , and  $1$ , with no switches from  $+1$  to  $-1$  or  $-1$  to  $+1$ , and with  $2nm(1 + T\omega/\pi)$  discontinuities at most.*

<sup>1</sup>For the definition of controllability, see [2, Sect. 4-15].  $(A, B)$  is controllable iff  $\text{rank}[B, AB, \dots, BA^{n-1}] = n$ .

<sup>2</sup>A linear system that is controllable and with nonsingular  $A$  is called a *metanormal* system [6].

**Proof:** Theorem 1 combined with Theorem 3.2 of [6] gives the results.  $\square$

Let us consider a networked system where we should send the control signal  $\mathbf{u}(t)$  on time interval  $[kT, (k+1)T)$  at every sampling time  $kT$ ,  $k = 0, 1, 2, \dots$ . From Proposition 1, we use 1 bit for representing the change of the control values, and  $b$  bits for representing each switching time. From Theorem 1, we need in total

$$1 + 2nmb \left( 1 + \frac{T\omega}{\pi} \right) \text{ [bit]}$$

to represent the maximum hands-off control on time interval  $[0, T]$ , or

$$\frac{1}{T} + 2nmb \left( \frac{1}{T} + \frac{\omega}{\pi} \right) \text{ [bps]},$$

which is much smaller than representing a general signal on  $[0, T]$ . This is an advantage of the maximum hands-off control for networked control systems.

#### 4. CONCLUSION

In this presentation, we have introduced maximum hands-off control (the sparsest control) and shown that this control is equivalent to  $L^1$ -optimal control under some assumptions on the optimal control problem. The maximum hands-off control has a “bang-bang” property, which is very advantageous to networked control systems in view of compressed data representation.

#### REFERENCES

- [1] R. P. Aguilera, R. Delgado, D. Dolz, and J. C. Agüero, “Quadratic MPC with  $\ell_0$ -input constraint,” *IFAC 19th World Congress*, pp. 10888–10893, Aug. 2014.
- [2] M. Athans and P. L. Falb, *Optimal Control*, Dover Publications, 2007.
- [3] S. S. Chen, D. L. Donoho, and M. A. Saunders, “Atomic decomposition by basis pursuit,” *SIAM J. Sci. Comput.*, vol. 20, no. 1, pp. 33–61, Aug. 1998.
- [4] Y. C. Eldar and G. Kutyniok (eds), *Compressed Sensing*, Cambridge University Press, 2012.
- [5] H. Fawzi, P. Tabuada, and S. Diggavi, “Secure estimation and control for cyber-physical systems under adversarial attacks,” *IEEE Trans. Automatic Control*, vol. 59, no. 6, pp. 1454–1467, Jun. 2014.
- [6] O. Hájek, “ $L_1$ -optimization in linear systems with bounded controls,” *Journal of Optimization Theory and Applications*, vol. 29, no. 3, pp. 409–436, Nov. 1979.
- [7] K. Hayashi, M. Nagahara, and T. Tanaka, “A user’s guide to compressed sensing for communications systems,” *IEICE Trans. on Communications*, vol. E96-B, no. 3, pp. 685–712, Mar. 2013.
- [8] S. G. Mallat and Z. Zhang, “Matching pursuits with time-frequency dictionaries,” *IEEE Trans. Signal Processing*, vol. 41, pp. 3397–3415, Nov. 1993.
- [9] M. Nagahara and D. E. Quevedo, “Sparse representations for packetized predictive networked control,” *IFAC 18th World Congress*, pp. 84–89, Aug. 2011.
- [10] M. Nagahara, D. E. Quevedo, and D. Nešić, “Maximum hands-off control and  $L^1$  optimality,” *Proc. of 52nd IEEE CDC*, 2013.
- [11] M. Nagahara, D. E. Quevedo, and J. Østergaard, “Sparse packetized predictive control for networked control over erasure channels,” *IEEE Trans. Automatic Control*, vol. 59, no. 7, pp. 1899–1905, July 2014.

M. NAGAHARA IS WITH GRADUATE SCHOOL OF INFORMATICS, KYOTO UNIVERSITY, KYOTO 606-8501, JAPAN. (NAGAHARA@IEEE.ORG).